Εισαγωγή στις διαφορικές εξισώσεις [16] 5-12-2019

[B-53] $y'' + 8y' + 25y = 2\cos x$, $x \in \mathbb{R}$ || $\exists\ \alpha, \delta \in \mathbb{R}$

$\lim_{+\infty} [y(x) - a\cos(x-\delta)] = 0$

$y'' + 8y' + 25y = 0$

$\lambda^2 + 8\lambda + 25 = P(\lambda)$

$$\lambda_{1,2} = \frac{-8 \pm \sqrt{64-100}}{2} = \frac{-8 \pm i6}{2} = -4 \pm 3i.$$

$\{ e^{-4x}\cos 3x,\ e^{-4x}\sin 3x \}$
$x \in \mathbb{R}.$

$$W(x) = \begin{vmatrix} e^{-4x}\cos 3x & e^{-4x}\sin 3x \\ -4 \cdot e^{4x}\cos 3x - 3e^{-4x}\sin 3x & -4 \cdot e^{-4x}\sin 3x + e^{-3x}\cos 3x \end{vmatrix}$$

$= -4 \cdot e^{-8x}\cos 3x + \sin 3x + 3e^{-8x}\cos^2 3x + 4 \cdot e^{-8x}\cos 3x + \sin 3x$
$+ 3 \cdot e^{-8x}\sin 3x = 3 e^{-8x}$

$W(x) = 3 \cdot e^{-8x}$

$$W_1(x) = \begin{vmatrix} 0 & e^{-4x}\sin 3x \\ 1 & \end{vmatrix} = -e^{-4x}\sin 3x$$

$$W_2(x) = \begin{vmatrix} e^{-4x}\cos 3x & 0 \\ 0 & 1 \end{vmatrix} = e^{-4x}\cos 3x$$

$$y_1(x) = y_1(x)\int_0^x \frac{W_1(s)}{W(s)} b(s)\,ds + y_2(x)\int_0^x \frac{W_2(s)}{W(s)} b(s)\,ds$$

$$= e^{-4x}\cos 3x \int_0^x \frac{e^{-4s}\sin(3s)}{e^{-8s}} 2\cos(s)\,ds + e^{-4s}\sin 3x \int_0^x \frac{e^{4s}\cos(3s)}{e^{8s}} 2\cos(s)\,ds$$

$$= \frac{1}{3} e^{-4x} \cos 3x \int_0^x e^{4s} \sin(3s)\, 2\cos(s)\, ds + e^{-4x} \sin 3x \int_0^x e^{4s} \cos 3x\, 2\cos(s)\, ds$$

$$y_p(x) = \frac{1}{3} \cdot e^{-4x} \int_0^x e^{4s}\, 2\cos(s) \underbrace{\left[-\cos(3x)\sin(3s) + \sin(3x)\cos(3s)\right]}_{\sin(3x-3s)} ds$$

$$y_p(x) = \frac{1}{3} e^{-4x} \int_0^x 2 \cdot e^{4s} \cos s \sin(3x-3s)\, ds \quad ①$$

$$\boxed{\sin A \cos B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]}$$

$$① \Rightarrow y_p(x) = \frac{1}{3} \cdot e^{-4x} \cdot \int_0^x 2 \cdot e^{s} \frac{1}{2} \cdot \sin(3x - 2s) + \sin(3x - 4s)\Big]$$

$$= \frac{1}{3} e^{-4x} \left[A \cdot e^{4x} \sin b(x) + B\, e^{4x} \cos b(x)\right]$$

$$A \sin(bx) + B\cos(bx)$$
$$= A\left[\sin(bx) + \frac{3}{4}\cos(bx)\right]$$
$$= A\left[\sin(bx) + \frac{\sin\vartheta}{\cos\vartheta} \cos b(x)\right]$$

$$\frac{A}{\cos\vartheta}\left[\underbrace{\sin(bx)\cos\theta + \sin(\vartheta)\cos(bx)}_{\sin(bx+\vartheta)}\right.$$

$\uparrow a$

$$\boxed{\left[e^{(k+\lambda i)x}\right]' = (k+\lambda i) \cdot e^{(k+\lambda i)x}, \quad (e^{\lambda x})' = \lambda e^{\lambda x}, \ \lambda \in \mathbb{C}}$$

{B-2.1 λυμ.} $y'' + a_1 y' + a_0 y = b(x)$

$\alpha_0, \alpha_1 (\in \mathbb{R})$ $\lambda^2 + \alpha_1 \lambda + \alpha_0$ (χπ)

$\hookrightarrow Re\lambda_1 < 0$

$Re\lambda_2 < 0$

b φραγμ. $\Rightarrow$ όλες οι λύσεις είναι φραγμένες.

β.σ. $\{e^{\lambda_1 x}, e^{\lambda_2 x}\}_{x \in \mathbb{R}}$

1η περίπτωση: $\lambda_1, \lambda_2 \in \mathbb{R} \to \{e^{\lambda_1 x}, e^{\lambda_2 x}, x \in \mathbb{R}\}$.

$\lambda_1, \lambda_2 \in \mathbb{C} \to \begin{Bmatrix} e^{Re\lambda_1 x} \cos Im\lambda_1 x \\ e^{Re\lambda_2 x} \sin Im\lambda_1 x \end{Bmatrix}$.

$$w(x) = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} \end{vmatrix} = (\lambda_2 - \lambda_1) e^{(\lambda_1 + \lambda_2) x}$$

$$w_1(x) = -e^{\lambda_2 x}$$

$$w_2(x) = e^{\lambda_1 x}$$

$$y_\mu(x) = e^{\lambda_1 x} \int_0^x \frac{-e^{\lambda_2 s}}{(\lambda_2 - \lambda_1) e^{(\lambda_2 + \lambda_1) s}} b(s) ds + e^{\lambda_2 x} \int_0^x \frac{e^{\lambda_1 s}}{(\lambda_2 - \lambda_1) e^{(\lambda_1 + \lambda_2) s}} b(s) ds$$

$$y_\mu(x) = \frac{1}{\lambda_1 - \lambda_2} \left[e^{\lambda_1 x} \int_0^x \frac{b(s)}{e^{\lambda_1 s}} + e^{\lambda_2 x} \int_0^x \frac{b(s)}{e^{\lambda_1 s}} ds \right].$$

$$\boxed{|e^{(a+bi)}| = |e^a (\cos b + i \sin b)|}$$

$$|e^{\lambda_1 x}| = e^{(Re\lambda_1) x}$$

$$|e^{\lambda_1 x}| = e^{(Re\lambda_1) x} \le 1, \ x \ge 0$$

$$|e^{(-5+2i) x}| = e^{-5x} \le 1, \ x \ge 0.$$

$\bullet\ |y_\mu(x)| \le \frac{1}{|\lambda_1-\lambda_2|}\left[\underset{e^{\lambda_1 x}}{|} \int_0^x |b(s)| \, |e^{-\lambda_1 s}| ds \right.$

$+ |e^{\lambda_2 x}| \int_0^x |b(s)| \, |e^{-\lambda_2 s}| ds \left. \right]$.

$|e^{\lambda_1 x}| \int_0^x |b(s)| \, |e^{-\lambda_1 s}| ds = e^{(Re\lambda_1)x} \int_0^x |b(s)| e^{-(Re\lambda_1)s} ds$

$\le M e^{(Re\lambda_1)x} \int_0^x e^{-(Re\lambda_1)s} ds$

$= M e^{(Re\lambda_1)x} \frac{e^{-(Re\lambda_1)s}}{-Re\lambda_1} \Big|_0^x = M \frac{e^{(Re\lambda_1)x}}{-Re\lambda_1} [e^{-(Re\lambda_1)x} -$

$\frac{M}{-Re\lambda_1} [e^0 - e^{(Re\lambda_1)x}] \le \frac{M}{-Re\lambda_1}$ //

Επομένως η μερική λύση είναι φραγμένη

$\boxed{B-39}$ $y^{(n)} + \alpha_{n-1} y^{(n-1)} + \dots + \alpha_1 y' + \alpha_0 y = b$, $\alpha_i \in \mathbb{R}$, $b \in C(\mathbb{R})$

y_1 λύση της (E_0) που πληροί τις αρχικές συνθήκες
$y_0(0) = 0 = \dots = y_0^{(n-1)}(0)$, $y_0^{(n-1)}(0) = 1$

$\rightarrow y_\mu(x) = \int_0^x y_0(x-s) b(s) ds$ λύση της (E) με
$y^{(i)}(0) = 0$, $i = 0, \dots, n-1$.

α_0 $y_\mu(x) = \int_0^x y_0(x-s) b(s) ds$

α_1 $y_\mu'(x) = \overset{\to 0}{y_0(x-x)} b(x) + \int_0^x y_0'(x-s) b(s)$

α_2 $y_\mu''(x) = \overset{\to 0}{y_0'(x-x)} b(x) + \int_0^x y_0''(x-s) b(s) ds$

α_{n-1} $y_\mu^{(n-1)}(x) = y_0^{(n-2)}\cancel{(x-x)}b(x) + \int_0^x y_0^{(n-1)}(x-s)b(s)\,ds$

$1=\alpha_n$ $y_\mu^{(n)}(x) = y_0^{(n-1)}(x-x)\,b(x) + \int_0^x y_0^{(n)}(x-s)b(s)\,ds$

- $L(y_\mu) = \alpha_0\int_0^x y_0(x-s)b(s)ds + \alpha_1\int y_0'(x-s)b(s)ds +$

$+ \dots + \alpha_{n-1}\int_0^x y_0^{(n-1)}(x-s)b(s)ds + \dots + b(x) + \int_0^x y_0^{(n)}(x-s)$
$b(s)ds$

$= b(x) + \int_0^x b(s)[\alpha_0 y_0(x-s) + \alpha_1 y'(x-s) + \dots + \alpha_{n-1}y^{(n-1)}(x) + y_n(x-s)]$
$ds.$

$= b(x)$

aduces $\boxed{B-28}$ $y'' + 2ay' + \omega^2 y = C\sin(\omega x)$ $\quad 0 < a \le \omega,\ c > 0$

$y'' + 2ay' + \omega^2 y = 0$

$\lambda^2 + 2a\lambda + \omega^2$, $\lambda_{1,2} = \dfrac{-2\alpha \pm \sqrt{4a^2 - 4\omega^2}}{2} = -a \pm \sqrt{a^2 - \omega^2}$

$= -a \pm \delta i$, $\delta = \sqrt{|\alpha^2 - \omega^2|}$

$\{e^{-\alpha x}\cos\delta x,\ e^{-\alpha x}\sin\delta x\}$

$W(x) = \begin{vmatrix} e^{-\alpha x}\cos\delta x & e^{-\alpha x}\sin\delta x \\ -ae^{-\alpha x}\cos\delta x - \delta\cdot e^{-\alpha x}\sin\delta x & -\alpha\cdot e^{\alpha x}\sin\delta x + \delta\cdot e^{-\alpha x}\cos\delta x \end{vmatrix}$

$= \delta\cdot e^{2\alpha x} \neq 0$

$W_1(x) = -e^{-\alpha x}\sin\delta x$

$W_2(x) = e^{-\alpha x}\cos\delta x.$

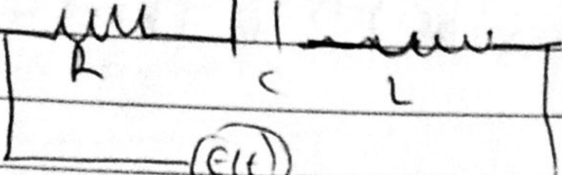
$$y_\mu(x) = e^{-\alpha x}\cos\delta x \int_0^x \frac{-e^{-\alpha s}\sin(\delta s)}{\delta e^{-\alpha s}}\, c\sin(\omega s)\,ds$$

$$+ e^{-\alpha x}\sin\delta x \int_0^x \frac{e^{-\alpha s}\cos(\delta s)}{\delta \cdot e^{-\alpha s}}\, c\sin(\omega s)\,ds$$

$$= e^{-\alpha x}\cos(\delta x)\int_0^x - e^{as}\cos(\delta s)\,ds + e^{-\alpha x}\sin\delta x\int_0^x e^{-\alpha s}\cos(\delta s)\, c\sin(\omega s)\,ds.$$

Εφαρμογή:

1)



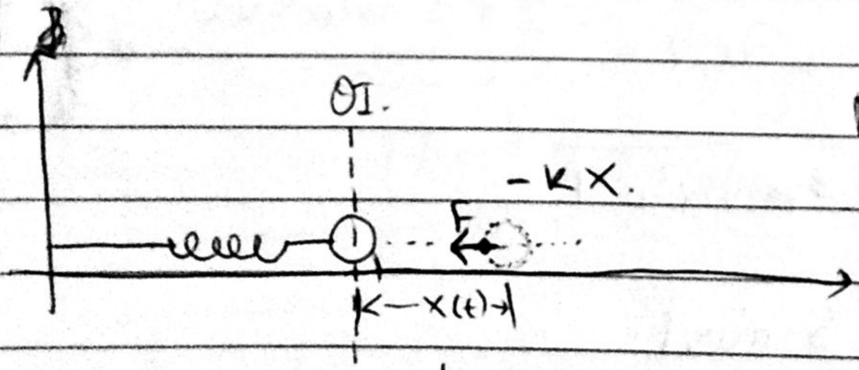
$$RI(t) + \frac{1}{L}Q(t) + C\frac{dI(t)}{dt} = E(t)$$

$$RI'(t) + \frac{1}{L}Q'(t) + LI'(t) = E'(t).$$

$$I'(t) + \frac{R}{L}I'(t) + \frac{1}{CL}I(t) = \frac{E'(t)}{L}$$

$$\boxed{\{I''(t) + 2\gamma I'(t) + \omega_0^2 I(t) = f(t)\}}$$

2)



Ν.Ν.: $F = m\overset{x''(t)}{\gamma} = -kx$

$\Rightarrow mx'' + kx = 0.$

$\Rightarrow x'' + \frac{k}{m}x = 0.$

$\{\cos(\frac{k}{m}x)\}.$

Αν $F = -kx - bx'$

τότε $mx'' + bx' + kx = 0$. (εξ. σ. 2ας τάξης με σταθ. συντελ.)

B-56 $y'' + 2\gamma y' + \omega^2 y = f$, $\gamma, \omega > 0$

$\left.\begin{array}{l} \lim f(x) = L \neq 0 \\ f \in C(0,\infty) \end{array}\right\} \Rightarrow \lim_{x \to \infty} y(x) = \frac{L}{\omega^2}$, $L > 0$

$\gamma + \omega$

Εάν η f έχει όριο L τότε όλες οι λύσεις είναι σταθερού προσήμου (θετικού)

$\left\{\begin{array}{l} e^{-\gamma t}\cos(kt) = y_1 \to 0 \\ e^{-\gamma t}\sin(kt) = y_2 \to 0. \end{array}\right.$

$\lim_{t \to \infty} y_p(t) = \frac{L}{\omega^2}$

$$y_1(x)\int_0^x \frac{y_2(s)}{w(s)} f(s)\,ds$$

$$+ y_2(x)\cdot\int_0^x \frac{y_1(s)}{w(s)} f(s)\,ds$$

$\varepsilon > 0$

$f(x) \to c > 0 \Rightarrow \exists X_0 > 0 : |f(x) - L| < \varepsilon \; \forall x \geq X_0$

$\Rightarrow \{L - \varepsilon \leq f(x) \leq L + \varepsilon, \; x \geq X_0.\}$

$L - \varepsilon \leq f(s) < L + \varepsilon.$ ■

Σύστημα με σταθ. συντελεστές.

$y_1' = \alpha_1 y_1 + \alpha_2 y_2$ $\alpha_1, \alpha_2, b_1, b_2 \in \mathbb{R}$.

$y_2' = b_1 y_1 + b_2 y_2$

$y_1'' = \alpha_1 y_1' + \alpha_2 y_2' = \alpha_1(\alpha_1 y_1 + \alpha_2 y_2) + \alpha_2(b_1 y_1 + b_2 y_2)$

$\left\{\begin{array}{l} y_1' = \alpha y + \alpha y_2 \\ y_1'' = k y_1 + \lambda y_2. \end{array}\right. \to y_1'' + m y_1' + n y_1 = 0$

Σελ. 207.

Παράδειγμα $\begin{cases} y_1' = y_1 + 2y_2 \\ y_2' = 3y_1 + y_2 \end{cases}$

Λύση:

$$y_1'' = y_1' + 2y_2' = (y_1 + 2y_2) + 2(3y_1 + y_2)$$

$$\begin{cases} y_1'' = 7y_1 + 4y_2 \\ \text{κ}' \; y_1' = y_1 + 2y_2 \end{cases} \Rightarrow \begin{cases} y_1'' - 2y_1' = 5y_1 \Rightarrow \\ y_1'' - 2y_1' - 5y_1 = 0. \end{cases}$$

και μετά κάνουμε αντικατάσταση της y_1 σε μια από τις 2 εξισώσεις για να βρούμε την y_2 (αλγεβρικά)

$$y_2 = \frac{y_1' - y_1}{2}$$